Fortunate and unfortunate primes : Nearest primes from a prime factorial

by Cyril Banderier, Algorithms Project, INRIA.
September 1999 [Updated 12/18/2000]

On the distance between the prime factorial and the previous/next prime number

The prime factorial of order n is the product of the first n prime numbers: $f_n=p_1\,p_2\,...\,p_n$

Some interesting values of n are those for which f_n-1 or f_n+1 is prime.

A natural conjecture (we will explain after why) is the following one

The Prime Factorial Conjecture:

- The distance d_n^+ between the prime factorial f_n and the first prime greater than f_n is either 1, either a prime number.
- Similarly, the distance d_n^- between the prime factorial f_n and the first prime least than f_n is either 1, either a prime number.

The peculiar case of the first prime greater than a given prime factorial is studied in the section A2 of the famous book of Richard K. Guy "Unsolved Problems in Number Theory" (2nd edition, Springer, 1994) and is known as Fortune's Conjecture, which leads to the so-called "fortunate numbers".

Who was R.F. Fortune ?

Reo Franklin Fortune (1903-1979) was a social anthropologist, lecturer, social anthropology, Cambridge Univ. , specialist in Melanesian language and culture (confer the obituaries "Reo FORTUNE (1903-1979)", by Michael W. Young in "Canberra Anthropology" vol. 3, n.1, pp 105-108, 1980). R.F. Fortune joined in 1941 the Department of Anthropology at University of Toronto. Fortune was married to Margaret Mead from 1928 to 1935 (Margaret Mead did have three husbands...the first was Luther Cressman from 1923-28. I believe he was a minister. The second was Reo Fortune, a New Zealand psychologist turned anthropologist from 1928-35. The third was Gregory Bateson (1936-50). He was a British anthropologist whose strong natural science background influenced Mead's work. [Information from a book called Women Anthropologists - Selected Biographies edited by Gacs, Khan, McIntyre, and Weinberg. USA: Greenwood press, 1988]). Fortune is well known for his ethnographies of the Dobu and Manus islanders of the Pacific. What many anthropologists do not realize is that he is also known to mathematicians for his conjecture on prime numbers, or "Fortunate Numbers"! Levin et al. (1984), report a story that Fortune once attempted to conclude an academic dispute with McIlwraith by challenging him to a duel with any weapon of his choice from the collections of the Royal Ontario Museum. Here are other anecdotes kindly communicated to me by Richard L. Warms.

The reader amazed by this "link" between anthropology and mathematics should keep in mind that is not the only one : remember Claude Lévi-Strauss, whose mathematical considerations influenced his science and the structuralist school.

I am not aware of literature about the case of the first prime less than a given prime factorial, so I suggest to call "unfortunate numbers" the d_n^-'s (whereas the d_n^+'s are already known as fortunate numbers).

Heuristics: If d_n^+ were not prime, then $d_n^+=(p_{n+1})^2$ or a greater composite number.

Asymptocally, one has log(f_n)=O(p_n/log(p_n)) and Shinzel's conjecture (there is always a prime between x and x+ln(x)^2) thus implies that the d_n's are primes.

Computations: I give below the list of fortune and unfortunate numbers d_n for $n<1100$,
previous prime case (EIS A005235),
next prime case (EIS A055211).

(EIS refers to the Encyclopedia of integer sequences). Thus the 2 conjectures are checked for $n<1100$.

The prime factorial primes, aka as primorial primes (coined by Dubner) are the f_n+-1 with
n such that $p_1\, p_2 \ldots p_n+1$ is prime : n=1, 2, 3, 4, 5, 11, 75, 171, 172, 384, 457, 616, 643, 1391, 1613, 2122, 2647, 2673, 4413, 13494 [EIS A014545].
n such that $p_1\, p_2 \ldots p_n-1$ is prime : n=2, 3, 5, 6, 13, 24, 66 ,68, 167, 287, 310, 352, 564, 590, 620, 849, 1552, 1849 [EIS A057704].

The factorisation of the composite primorial numbers is given here.

References:

- C. Caldwell, Y. Gallot, On The Primality of n!+- 1 and 2.3.5...p+- 1, Math. Comput., 2000.
- H. Dubner, Factorial and primorial primes, J. Rec. Math., 19 (No. 3, 1987), 197-203.
- A. Fletcher, J. C. P. Miller, L. Rosenhead, and L. J. Comrie, An Index of Mathematical Tables. Vols. 1 and 2, 2nd ed., Blackwell, Oxford and Addison-Wesley, 1962, Vol. 1, p. 50.
- S. W. Golomb, The evidence for Fortune's conjecture, Math. Mag. 54 (1981), 209-210.
- P. Ribenboim, The Book of Prime Number Records. Springer-Verlag, NY, 2nd ed., 1989, p. 4.
- H. Ibstedt, A Few Smarandache Sequences, Smarandache Notions Journal, Vol. 8, No. 1-2-3, 1997, 170-183.